International Journal of Theoretical Physics, Vol. 34, No. 11, 1995

Some Examples of q-Regularization

Suemi Rodríguez-Romo[1]

Received February 18, 1995

An introduction to Hopf algebras as a tool for the regularization of relevant quantities in quantum field theory is given. We deform algebraic spaces by introducing q as a regulator of a noncommutative and noncocommutative Hopf algebra. Relevant quantities are finite provided $q \neq 1$ and diverge in the limit $q \to 1$. We discuss q-regularization on different q-deformed spaces for $\lambda\phi^4$ theory as examples to illustrate the idea.

1. INTRODUCTION

Thanks to the work done in expressing vector bundles, forms, integration, etc., on locally compact topological spaces X entirely in terms of the algebra $C(X)$ of complex continuous functions on X vanishing at infinity which forms a commutative C^* algebra, a generalization of ordinary geometry can be introduced. Namely, when expressed in terms of a C^* algebra the above notions make sense even when the C^* algebra is not commutative, therefore not of the form $C(X)$ (Connes, 1986). The simplest noncommutative geometries that have been studied are noncommutative and noncocommutative Hopf algebras, corresponding to both quantization and curvature.

Meanwhile in classical mechanics states are points of a manifold M and observables are functions on M; in the quantum case, states are one-dimensional subspaces of a Hilbert space H and observables are operators in H. Observables, in both classical and quantum mechanics, form an associative algebra, which is commutative in the classical case and noncommutative in the quantum case. So we can think of quantization as a procedure that replaces the classical algebra of observables by a noncommutative quantum algebra of observables. The noncommutative Heisenberg algebra, i.e., the algebra

[1] Centre of Theoretical Research, Facultad de Estudios Superiores Cuautitlán, UNAM, Unidad Militar, Cuautitlán Izcalli, Estado de México, 54768 México. E-mail: suemi@servidor.unam.mx.

0020-7748/95/1100-2179$07.50/0

that arises because momentum and space are not simultaneously measurable (Heisenberg uncertainty principle), is the best example to illustrate this idea. Generally speaking, it is expected that even using noncommutative geometry, one might nevertheless extend our regular notions of symmetry to the quantum world. If we consider the space of states endowed by a group structure, the functions on this are observables. To quantize such a system one has to construct a noncommutative associative algebra of functions on a locally compact topological group space; i.e., a quantum group (Drinfeld, 1986a,b, 1983a,b, 1985).

Thinking about quantization of the space-time metric itself, where we cannot use path integration techniques to express quantization in terms of classical fields, we claim the assumption of a smooth manifold structure for space-time to be meaningless in extremely small scales from the experimental viewpoint. The problem is that the finer the accuracy in the observation we ask for, the heavier the test particle we need; eventually the space-time curvature due to both the test particle and the space-time itself can be of the same magnitude. In this context, by relaxing the assumption of smoothness of the space-time manifold and introducing noncommutative algebraic geometry, we propose a scheme called q-regularization, so we can regulate relevant quantities in field theory before renormalizing. The parameter q ($q^2 \neq -1$) parametrizes the deformation to the noncommutative and noncocommutative framework in which relevant quantities in quantum field theories are finite for $q \neq 1$ and reduce to the unregulated, divergent, physical theory as $q \to 1$. Namely, as in dimensional regularization, we interpolate consistently to dimension $4 - \epsilon$, where the relevant quantities are finite (these would be infinite at dimension four); in q-regularization we extend relevant quantities in quantum field theory to a noncommutative and noncocommutative Hopf algebra or quantum group (by introducing the parameter q) where the relevant quantities are finite [these would be infinite at $q = 1$; i.e., in $C(X)$, the commutative limit].

We present two examples; the first one is constructed in a four-dimensional representation of a particular noncommutative space previously reported (Majid, 1990a). The second example is proposed having in mind q-spinors [two-dimensional objects with the generators of $A_q^{2/0}$, Manin's (1988) quantum plane, as entries]; constructed by the projective representation of the Heisenberg algebra, they are braided in a very specific way to obtain a q-deformed space.

The second example is intended as a first step to approach q-regularization in q-Minkowski space-time. We work out this example in a q-deformed space which can be related to both the first example's q-mutator algebra and previously reported (Carow-Watamura *et al.*, 1990, 1991) braided two copies of Manin's quantum planes. Since we do not impose reality conditions, among others, we are not working in any way in q-Minkowski space-time.

For the second example we want to learn more about the symmetries of our measure; we study a projection in the q-deformed space used and its relation to the $SU_q(2)$ measure. Moreover, we analyze the null directions of the corresponding Hopf algebra that lead to a q-deformed Galilei group.

This paper is organized as follows; in Section 2 we construct the Manin quantum plane out of the noncommutative Heisenberg algebra and introduce the q-spinors as a way to link the q-regularization scheme with physically meaningful concepts. In Section 3, we present two examples of q-regularization on q-deformed Euclidean spaces for $\lambda\phi^4$ theory. Our scheme can only be carried out in a very particular basis for functions defined on the q-deformed spaces chosen such that we end up with a Haar weight that reduces to an ordinary integration. Further work should be done to generalize this. Finally, in order to learn about desired properties of symmetry in this q-regularization we study the zero-time projection of the measure we have just introduced in the second example in terms of the $SU_q(2)$ measure and the null directions of the Hopf algebra that lead to a q-deformed Galilei group. The quantum Galilei group has been found as symmetry in condensed matter (Bonechi *et al.*, 1992a,b).

2. FROM HEISENBERG ALGEBRA TO q-SPINORS

The goal of this section is to link noncommutative Heisenberg algebra with two cocycles and q-spinors as defined by Manin (1988). Let us start with the fundamental Heisenberg commutator algebra generated by translations on phase space $(\mathbf{r}, \mathbf{p})$,

$$[r^i, p^j] = i\hbar\delta^{ij} \tag{1}$$
$$[r^i, r^j] = [p^i, p^j] = 0$$

We propose the following translation operator on phase space:

$$U(\mathbf{a}, \mathbf{b}) = e^{i(\mathbf{a}\cdot\mathbf{p}-\mathbf{b}\cdot\mathbf{r})/\hbar} \qquad \text{where} \quad \mathbf{a} \text{ and } \mathbf{b} \in R^n \tag{2}$$

In a ray or projective representation, equation (2) obeys the composition law (Djemi, 1992)

$$U(\mathbf{a}_2, \mathbf{b}_2)\cdot U(\mathbf{a}_1, \mathbf{b}_1) = e^{2\pi i\alpha_2\,(\mathbf{r};(\mathbf{a}_1,\mathbf{b}_1)(\mathbf{a}_2,\mathbf{b}_2))}\cdot U(\mathbf{a}_1 + \mathbf{a}_2, \mathbf{b}_1 + \mathbf{b}_2) \tag{3}$$

where $\mathbf{a}_1$, $\mathbf{b}_1$, $\mathbf{a}_2$, $\mathbf{b}_2 \in R^n$ and, for a free particle in quantum mechanics, the two cocycle α_2 for translations in the phase space is given by

$$2\pi\alpha_2(\mathbf{r}; (\mathbf{a}_1, \mathbf{b}_1), (\mathbf{a}_2, \mathbf{b}_2)) = \frac{1}{2\hbar}(\mathbf{a}_1\cdot\mathbf{b}_2 - \mathbf{a}_2\cdot\mathbf{b}_1) \tag{4}$$

Let us now consider the following infinitesimal Galilei transformation (Djemi, 1992):

$$\mathbf{r}' = \mathbf{r} + \mathbf{a}_1 = \mathbf{r} + \hbar\mathbf{u}, \qquad \mathbf{r}'' = \mathbf{r} + \mathbf{a}_2 = \mathbf{r} \tag{5}$$
$$\mathbf{p}' = \mathbf{p} + \mathbf{b}_1 = \mathbf{p}, \qquad \mathbf{p}'' = \mathbf{p} + \mathbf{b}_2 = \mathbf{p} + \hbar\mathbf{u}$$

where $\mathbf{u}$ is a unit vector in R^n.

If we define

$$q = e^{-i\hbar} \tag{6}$$

impose (5) as symmetry in equation (4), and substitute the result in equation (3), it is straightforward to prove that

$$U(\hbar\mathbf{u}, 0)U(0, \hbar\mathbf{u}) = qU(0, \hbar\mathbf{u})U(\hbar\mathbf{u}, 0) \tag{7}$$

is a realization of $A_q^{(2/0)}$; i.e., this fulfills the noncommutative algebra of Manin's (1988) quantum plane.

Like other authors (Carow-Watamura *et al.*, 1990, 1991), we call the following two-dimensional object a q-spinor (more properly, Weyl q-spinor):

$$Z^\rho = \begin{bmatrix} Z^1 \\ Z^2 \end{bmatrix} = \begin{bmatrix} U(\hbar\mathbf{u}, 0) \\ U(0, \hbar\mathbf{u}) \end{bmatrix}, \qquad \text{i.e.,} \quad \rho = 1, 2 \tag{8}$$

Below, in Example 2 we use an approach (Carow-Watamura *et al.*, 1990, 1991) in which the q-deformed space can be related to the tensor product representation of two q-spinor spaces called (Z^i, $\tilde{Z}^i$). A pair of q-spinors ($i = 1, 2$) is introduced in each space. Hereafter Greek indices are for spinor subscripts and Roman indices for different spinors. We also require the following braiding:

$$Z^i\tilde{Z}^j = \hat{R}^{ij}_{j'i'}\tilde{Z}^{j'}Z^{i'} \tag{9}$$

where $\hat{R}^{ij}_{j'i'}$ is the Yang–Baxter matrix for $SL_q(2, C)$.

3. EXAMPLES OF q-REGULARIZATION

In this section we present two examples of q-regularization for $\lambda\phi^4$ theory on two apparently different q-deformed spaces, both Euclidean. The first case involves a four-dimensional version of a Hopf algebra previously reported (Majid, 1990a); we propose to extend momenta internal to Feynman loops to a noncommutative structure. The second example involves a braided 4-dimensional representation of Manin's quantum plane (so-called q-spinors) where some particular transformations on the generators of this Hopf algebra relate to the one used in Example 1. Actually, Example 1 is posed in order to better explain Example 2, which is considered as a preliminary step for formulating q-regularization on q-Minkowski space-time.

Example 1. From Majid (1990a), let us consider the Hopf algebra L generated by (l_1, l_2, l_3, l_4) and

$$[l_k, l_j] = il_jQ' \qquad \text{for} \quad k = 2, 4 \text{ and } j = 1, 3 \tag{10}$$

where $Q' = (1 - q)^{1/2}$. Define on this the antipode map as

$$S(l_k) = -l_k, \qquad S(l_j) = -q^{-i}l_jq^{-l_k/Q'} \tag{11}$$

The coproduct map is given by

$$\Delta l_k = l_k \otimes 1 + 1 \otimes l_k, \qquad \Delta l_j = l_j \otimes 1 + q^{l_k/Q'} \otimes l_j \tag{12}$$

and the counit is

$$\epsilon(l_k) = \epsilon(l_j) = 0 \tag{13}$$

Additionally, L can become a C^* algebra if we define

$$l_k^* = l_k, \qquad l_j^* = l_j^* q^{i/2} \tag{14}$$

For every finite-dimensional Hopf algebra there is an invariant integration, the Haar weight $\int$, unique up to normalization.

A basis

$$B^{a_1 \cdots a_4} = e^{ia_1l_1} \cdots e^{ia_4l_4}$$

where $a_n \in C$, with C complex, is chosen; then the dual basis $D_{a'_1 \cdots a'_4}$ is given via

$$B^{a_1 \cdots a_4}D_{a'_1 \cdots a'_4} = \delta(a'_1 - a_1) \cdots \delta(a'_4 - a_4), \qquad a'_n \in C \tag{15}$$

where the Dirac delta functions δ have been defined with respect to the usual Lebesgue integration; then it is straightforward, by analogy with the case of finite-dimensional Hopf algebras (Larson and Radford, 1988), to prove (Rodríguez-Romo, 1994)

$$\iint f = [2\pi\delta(0)]^k \int \prod_j da'_j\, f'(0, a'_j(1 - q^{-i})) \tag{16}$$

for all j and f suitable of being written on the basis $B^{a_1 \cdots a_4}$. In the case $q \neq 1$ and assuming proper analycity and decay of f' (the Fourier transform of the Wick ordered function f), (16) might be finite for suitable f. If $q = 1$, equation (16) certainly diverges.

We propose, from $\lambda\phi^4$ theory, to q-regularize the vertex corrections with contributions given by

$$\Gamma(s) = \frac{(-i\lambda)^2}{2} \iint \frac{d^4 l}{(2\pi)^4} \frac{i}{(l-p)^2 - \mu_0^2 - i\epsilon} \frac{i}{l^2 - \mu_0^2 + i\epsilon} \tag{17}$$

where s is any Mandelstam variable. These corrections diverge logarithmically.

Let us extend the internal momentum in the Feynman loop in $\Gamma(s)$ to the noncommutative algebraic framework by considering instead of the standard Lebesgue integration the Haar weight above defined on the basis $B^{a_1 \cdots a_4}$, thus:

$$\Gamma_q(s) = \frac{\lambda_0^2 \delta(0)}{2(2\pi)^3} \times \int \frac{d^j l_j'}{[p_k^2 + (l_j'(1-q^{-i}) - p_j)^2 - \mu_0^2 + i\epsilon][(l_j'(1-q^{-i}))^2 - \mu_0^2 + i\epsilon]} \tag{18}$$

where l_j' are the odd components of the dual internal momentum that was extended to noncommutative geometry and p_k (p_l) are the even (odd) components of the external momentum in standard Euclidean commutative four-dimensional space-time. Unless $q = 1$, (18) is finite; thus we have a regularization scheme. An additional attempt at q-renormalization has recently been presented (Rodríguez-Romo, 1994). Since we extend to the noncommutative framework only the internal momentum degrees of freedom, the lack of locality resulting from this extension has no experimental consequences in this case (Fredenhagen, 1981).

Example 2. Let us consider the Hopf algebra H generated by 1 and $(a, \bar{a}, b, \bar{b})$ such that

$$\begin{aligned} &[b, \bar{b}] = 0, && [a, \bar{a}] = 2(q^{-1} - q)q^{(1/2Q')(b+3\bar{b})} \\ &[\bar{b}, a] = [b, a] = 2Q'\bar{a}, && [\bar{b}, \bar{a}] = [b, \bar{a}] = 2Q'a \end{aligned} \tag{19}$$

The coproduct map Δ in this Hopf algebra is

$$\begin{aligned} &\Delta a = a \otimes 1 + q^{b/Q'} \otimes a, && \Delta b = b \otimes 1 + 1 \otimes b \\ &\Delta \bar{a} = \bar{a} \otimes 1 + q^{\bar{b}/Q'} \otimes \bar{a}, && \Delta \bar{b} = \bar{b} \otimes 1 + 1 \otimes \bar{b} \end{aligned} \tag{20}$$

the antipode map S is

$$\begin{aligned} S(a) &= \frac{1}{2}\{-(q^{-2} + q^2)aq^{-b/Q'} + (q^2 - q^{-2})\bar{a}q^{-b/Q'}\} \\ S(\bar{a}) &= \frac{1}{2}\{(q^2 - q^{-2})aq^{-\bar{b}/Q'} - (q^{-2} + q^2)\bar{a}q^{-\bar{b}/Q'}\} \\ S(b) &= -b, \qquad S(\bar{b}) = -\bar{b} \end{aligned} \tag{21}$$

and finally the counit map ϵ is

$$\epsilon(a) = \epsilon(\overline{a}) = \epsilon(b) = \epsilon(\overline{b}) = 0 \tag{22}$$

Furthermore, we can make this into a *-algebra via

$$b^* = b, \qquad \overline{b}^* = \overline{b}, \qquad a^* = aq^{i/2}, \qquad \overline{a}^* = \overline{a}q^{i/2}$$

iff q is a primitive root of unity such that $q^4 = 1$.

We would like to relate H with

$$X^{ij} = \tilde{Z}^i Z^j \in A_q^{2/0} \otimes A_q^{2/0}, \qquad i, j = 1, 2 \tag{23}$$

where $\tilde{Z}^i$ and Z^j were introduced in Section 2 [Equations (8) and (9)]. It is straightforward to prove that $A_q^{2/0} \otimes A_q^{2/0}$ is isomorphic to the real algebra generated by 1 and $(A, \overline{A}, B, \overline{B})$, where

$$A = X + Y, \qquad \overline{A} = X - Y, \qquad B = Z + T, \qquad \overline{B} = Z - T \tag{24}$$

and

$$\begin{aligned} X &= q^{-1/2}X_{11}, \qquad Y = q^{-1/2}X_{12} \\ Z &= \frac{q^{-1}X_{21} - qX_{22}}{(q + q^{-1})^{1/2}}, \qquad T = \frac{X_{21} + X_{22}}{q(q + q^{-1})^{1/2}} \end{aligned} \tag{25}$$

To relate H to $A_q^{2/0} \otimes A_q^{2/0}$, let us rewrite the $(A, \overline{A}, B, \overline{B})$ generators, for $q \neq 1$, as follows:

$$\begin{aligned} A &= a, \qquad \overline{A} = \overline{a} \\ B &= q^{b/Q'}, \qquad \overline{B} = q^{\overline{b}/Q'} \end{aligned} \tag{26}$$

On the other hand, it is straightforward to prove that in H, $((a + \overline{a}), (a - \overline{a}), b, \overline{b})$ corresponds to the algebra L with generators (l_1, l_2, l_3, l_4) defined in Example 1. If $q \to 1$, the algebra becomes the commutative algebra of functions on the space generated by $(a, \overline{a}, b, \overline{b})$ and the unit.

As in Example 1, we proceed by defining the Haar measure $\int\int$ as a map $H \to C$ such that

$$\iint f = f_{(1)} \iint f_{(2)}, \qquad \forall f \in H \tag{27}$$

Here we have expressed the action of Δ on f as $\Delta f = f_{(1)} \otimes f_{(2)}$. We remark that it is well known in the theory of Hopf algebras (Majid, 1990b) that (27) is the dual formulation of the usual left invariance.

By analogy with the case of finite-dimensional Hopf algebras (Larson and Radford, 1988), we use the following formal expression for (27):

$$\iint f = \mathrm{Tr}_H\, L_f S^2 \tag{28}$$

where L_f stands for f acting by left multiplication on H.

From (21) it follows that

$$\begin{gathered} S^2(a-\bar{a}) = w^{-1}(a-\bar{a}); \qquad S^2(a+\bar{a}) = w^{-1}(a+\bar{a}) \\ S^2 b = b; \qquad S^2\bar{b} = \bar{b} \end{gathered} \tag{29}$$

where $w^{-1} = f(q)$ and $\lim_{q\to 1} w^{-1} = 1$. This shall be used below.

To compute $\iint$ we propose the following basis in H:

$$\begin{aligned} F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6} &= (F^{\lambda_1\lambda_2}, F^{\lambda_3\lambda_4}, F^{\lambda_5\lambda_6}) \\ &= (e^{i\lambda_1\bar{b}}e^{i\lambda_2(a-\bar{a})/2}, e^{i\lambda_3\bar{b}}e^{i\lambda_4(a+\bar{a})/2}, e^{i\lambda_5\bar{b}}e^{i\lambda_6 b}) \end{aligned} \tag{30}$$

where

$$F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6} \in H \qquad \text{and} \qquad (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \in R$$

We associate to $F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6}$ a dual basis

$$F_{\lambda_1'\lambda_2',\lambda_3'\lambda_4',\lambda_5'\lambda_6'} \in (A_q^{2/0} \otimes A_q^{2/0})^!$$

where $(A_q^{2/0} \otimes A_q^{2/0})^!$ is the dual Hopf algebra of $A_q^{2/0} \otimes A_q^{2/0}$, such that

$$\begin{aligned} &F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6}F_{\lambda_1'\lambda_2',\lambda_3'\lambda_4',\lambda_5'\lambda_6'} \\ &\quad = (\delta(\lambda_1'-\lambda_1)\delta(\lambda_2'-\lambda_2), \delta(\lambda_3'-\lambda_3)\delta(\lambda_4'-\lambda_4), \delta(\lambda_5'-\lambda_5)\delta(\lambda_6'-\lambda_6)) \end{aligned} \tag{31}$$

In the basis (30) we have introduced six parameters λ_i, one for each generator involved. They are dual variables to the noncommutative parameter.

Theorem 1. The Haar weight $\iint F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6}$ defined in (28), for a basis $F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6}$ chosen as in (30), reduces to an ordinary integration.

Proof. From (19) we know that

$$\begin{aligned} [\bar{b}, (a-\bar{a})] &= -2Q'(a-\bar{a}) \\ [\bar{b}, (a+\bar{a})] &= 2Q'(a+\bar{a}) \\ [\bar{b}, b] &= 0 \end{aligned}$$

Note that $(a+\bar{a})/2 = X$, $(a-\bar{a})/2 = Y$ in (25). Substituting the basis given by equation (30) in (28) and using the Glaube formula for operators, we obtain the ordinary integral

$$\iint F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6}$$

$$= \left(\int_{-\infty}^{\infty} d\lambda_1'\, d\lambda_2'\, \delta(\lambda_1' - (\lambda_1 + \lambda_1'))\delta(\lambda_2' - (\lambda_2 e^{-2i\lambda'_1 Q'} + w^{-1}\lambda_2')),\right.$$

$$\int_{-\infty}^{\infty} d\lambda_3'\, d\lambda_4'\, \delta(\lambda_3' - (\lambda_3 + \lambda_3'))\delta(\lambda_4' - (\lambda_4 e^{2i\lambda'_3 Q'} + w^{-1}\lambda_4')),$$

$$\left.\int_{-\infty}^{\infty} d\lambda_5'\, d\lambda_6'\, \delta(\lambda_5' - (\lambda_5 + \lambda_5'))\delta(\lambda_6' - (\lambda_6 + \lambda_6'))\right) \tag{32}$$

QED

The basis $F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6}$ in equation (30) admits an expression in terms of the q-spinor defined in (8) out of the projective representation for the Heisenberg algebra. Furthermore, this basis can be rewritten in terms of q-Majorana spinors built using q-Weyl spinors in analogy with the commutative algebraic formulation. As a result of this we can show that it does not matter if we think in terms of integrating out noncommutative light-cone coordinates, Weyl q-spinors, or Majorana q-spinors degrees of freedom; the result is exactly the same. Furthermore, the Haar measure $\iint$ defined on H can be written in terms of ordinary integration.

Theorem 2. For a suitable $f \in H$ that can be expressed on the basis $F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6}$ given in (30) (or any of their different q-spinor representations), $\iint f$ as defined in (28) contains a component that can be q-regularized, i.e., is finite provided $q \neq 1$, but infinite in the limit $q = 1$.

Proof. It is straightforward to show that for any function f defined on H with basis $F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6}$ the following transformation holds:

$$f = :f': = \int_{-\infty}^{\infty} d\lambda_1\, d\lambda_2\, \tilde{f}(\lambda_1, \lambda_2) F^{\lambda_1\lambda_2} + \int_{-\infty}^{\infty} d\lambda_3\, d\lambda_4\, \tilde{f}(\lambda_3, \lambda_4) F^{\lambda_3\lambda_4} + \int_{-\infty}^{\infty} d\lambda_5\, d\lambda_6\, \tilde{f}(\lambda_5, \lambda_6) F^{\lambda_5\lambda_6} \tag{33}$$

where we express f as a normal ordered form of f', in terms of the generators, namely, putting $\bar{b}$ to the left of a, $\bar{a}$, and b in the light-cone coordinate approach; $\{\sigma^3, \sigma^0\}$ to the left of $\{\sigma^+, \sigma^-\}$ in the Weyl q-spinor formulation; and finally $\{\gamma^3, \gamma^0\}$ to the left of $\{\gamma^1, \gamma^2\}$ in the Majorana q-spinor basis.

Here $(\sigma^0, \sigma^3, \sigma^+, \sigma^-)$ and $(\gamma^0, \gamma^1, \gamma^2, \gamma^3)$ are q-deformed Pauli and Dirac matrices (Carow-Watamura *et al.*, 1991). Additionally, $\tilde{f}$ is the Fourier transform of f', i.e.,

$$\tilde{f}(\lambda_i, \lambda_j) = (2\pi)^{-2} \int_{-\infty}^{\infty} d\mu_i \, d\mu_j \, f'(\mu_i \mu_j) e^{-i\mu_i\lambda_i} e^{-i\mu_j\lambda_j}$$

$$i, j = (1, 2), (3, 4), (5, 6) \tag{34}$$

Then carrying out integration on $\lambda_1, \lambda_3, \lambda_5, \lambda_6$, we obtain

$$\begin{aligned}\iint f = &\int_{-\infty}^{\infty} d\lambda'_1 \, d\lambda'_2 \, d\lambda_2 \; \tilde{f}(0, \lambda_2)\delta(\lambda'_2(1 - w^{-1}) - \lambda_2 e^{-2i\lambda'_1 Q'}) \\ &+ \int_{-\infty}^{\infty} d\lambda'_3 \, d\lambda'_4 \, d\lambda_4 \; \tilde{f}(0, \lambda_4)\delta(\lambda'_4(1 - w^{-1}) - \lambda_4 e^{-2i\lambda'_3 Q'}) \\ &+ \int_{-\infty}^{\infty} d\lambda'_5 \, d\lambda'_6 \; \tilde{f}(0, 0)\end{aligned} \tag{35}$$

which, after changing the order of integration and integrating on λ_2 and λ_4, becomes

$$\begin{aligned}\iint f = &\int_{-\infty}^{\infty} d\lambda'_1 \, d\lambda'_2 \; e^{\lambda'_1 Q'} \tilde{f}(0, \lambda'_2(1 - w^{-1}) e^{2i\lambda'_1 Q'}) \\ &+ \int_{-\infty}^{\infty} d\lambda'_3 \, d\lambda'_4 \; e^{-\lambda'_3 Q'} \tilde{f}(0, \lambda'_4(1 - w^{-1}) e^{-2i\lambda'_3 Q'}) \\ &+ \int_{-\infty}^{\infty} d\lambda'_5 \, d\lambda'_6 \; \tilde{f}(0, 0)\end{aligned} \tag{36}$$

The last term in equation (36) corresponds to the ordinary divergent term that appears in the standard commutative algebraic formulation of quantum field theory; there is no way we can recover a finite term out of this in the limit $q \to 1$. Checking the noncommutative Hopf algebra generated by X^{ij}, we find why this is so; T is central with respect to (X, Y, Z), so this part of the Haar measure is not really defined on a noncommutative algebraic variety. Therefore we can extract out of $\iint f$ a q-regularizable part

$$\begin{aligned}&\iint f - \int_{-\infty}^{\infty} d\lambda'_5 \, d\lambda'_6 \tilde{f}(0, 0) \\ &= [2\pi\delta(0)]\left\{\int_{-\infty}^{\infty} d\lambda'_2 \tilde{f}(0, \lambda'_2(1 - w^{-1})) \right.\\ &\qquad \left. + \int_{-\infty}^{\infty} d\lambda'_4 \tilde{f}(0, \lambda'_4(1 - w^{-1})\right\}\end{aligned} \tag{37}$$

But $\lim_{q\to 1} w^{-1} = 1$; thus, as $q \to 1$, $\int\int f - \int_{-\infty}^{\infty} d\lambda_5' \, d\lambda_6' \, \tilde{f}(0, 0)$ diverges; in contrast, at $q \neq 1$ and assuming suitable analycity and decay of $\tilde{f}$ to allow contour integration, (37) can be made finite for suitable f; moreover, this is proportional to $(1 - w^{-1})^{-1}$.

In the limit $q \to 1$, the transformation described in (26) is nonsense, because in this limit the map $H \to L$ is singular. We remark that this does not mean that the q-regularization scheme performed on the algebra H and described up to here is lacking sense in the case $q = 1$, but only the map that relates this with $A_q^{2/0} \otimes A_q^{2/0}$. We are interested in this map because it might be of some help in the future construction of q-regularization on q-Minkowski space-time. Further work needs to be done in this direction. QED

For obvious reasons, the vertex correction for $\lambda\phi^4$ theory described in Example 1 is suitable for being q-regularized on H just as in Example 2 it was on L. Further work should be done to generalize these examples to more interesting cases. Since this scheme is strongly basis dependent, a complete analysis of the class of functions suitable for being q-regularized on physically interesting bases is needed. Note that q-regularization may be considered equivalent to dimensional regularization in a similar sense to the McKane (1980) and Parisi and Sourlas (1979, 1980) case.

4. COMMENTS AND REMARKS

In the paper wherein Woronowicz (1987) proves the existence and uniqueness of the Haar measure, i.e., the unique state invariant under left (and simultaneously right) shifts, for any compact quantum group, he proposes the following q-integration on $SU_q(2)$:

$$\int_{q^0}^{1} \mathbf{f} = (1 - q) \sum_{k=0}^{\infty} q^k \mathbf{f}(q^k) \qquad \text{for any} \quad \mathbf{f} \in SU_q(2) \tag{38}$$

On the other hand, let us set $T = 0$ in (25); from equation (7), we get

$$U_2(\hbar\mathbf{u}, 0) = q^{1/2} \tilde{u}_1(0, \hbar\mathbf{u}), \qquad U_2(0, \hbar\mathbf{u}) = -q^{-1/2} \tilde{u}_1(\hbar\mathbf{u}, 0) \tag{39}$$

This is equivalent to setting $\tilde{Z}^\rho = \epsilon^{\rho\sigma} \overline{Z}_\sigma$ in equation (9). Thus

$$A = X + Y, \qquad \overline{A} = X - Y, \qquad B = \overline{B} = Z$$

We write $X^1 = \frac{1}{2}(A + \overline{A})$, $X^2 = \frac{1}{2}(A - \overline{A})$, and $X^3 = B$; then in terms of this (X^1, X^2, X^3) 3-dimensional vector representation, we propose the following basis:

$$F^{\lambda_1\lambda_2,\lambda_3\lambda_4} = (e^{i\lambda_1 x^3} e^{i\lambda_2 x^1}, \ e^{i\lambda_3 x^3} e^{i\lambda_4 x^2}) \tag{40}$$

where $X^1 = x^1$, $X^2 = x^2$, and $X^3 = q^{x^3/Q'}$, as was done in (26).

From work on the category of representations of a Hopf algebra we can write the action of any function f of the Hopf algebra $SU_q(2)$ on its vector representation V through the corresponding basis,

$$\mathbf{f} \cdot e^j_m = \sum_{i=+,-,0} f^i \mathbf{e}_i \cdot e^j_m \in V, \qquad \forall \mathbf{f} \in SU_q(2) \tag{41}$$

where $\mathbf{e}_+ = X^+$, $\mathbf{e}_- = X^-$, $\mathbf{e}_0 = H$ is the $SU_q(2)$ basis and $f^i \in C$.

From equation (41) it is clear that the Woronowicz map $\int \mathbf{f} \to C$ for the $SU_q(2)$ Haar measure induces a $\int\int f \to C$ map for the vector representation of the Hopf algebra $SU_q(2)$, inducing another for Λ, which is written in terms of $SU_q(2)$. We define the Λ matrix as

$$\Lambda^{(ij)}_{(i'j')} \equiv \tilde{M}^i_{i'} M^j_{j'}; \qquad M \in SL_q(2, C), \quad \tilde{M} \in \tilde{S}L_q(2, C) \tag{42}$$

Thus, the similarity of equations (34) and (38) can be understood in these terms.

From these two facts and the obvious similarity of the q-integration carried out in equation (34) and the one depicted in equation (38), we think that the q-time zero projection in the q-deformed space defined for the second example corresponding to the $\tilde{M}^\dagger = M^{-1}$ identification reduces $\int\int f - \int_{-\infty}^{\infty} d\lambda'_5 \, d\lambda'_6 \tilde{f}(0, 0)$ in Theorem 2 to the Haar weight on the vector representation of Λ written in terms of $SU_q(2)$.

Finally, we show how null directions in Λ can lead to the quantum mechanical Galilei group. By imposing the null bi-ideals

$$u^1_2 = 0, \qquad u^2_1 = 0$$

$$u^1_1 u^2_2 = u^2_2 u^1_1 = 1$$

with $u^i_j \in M$ (the same hold for $\tilde{u}^i_j$, with $\tilde{u}^i_j \in \tilde{M}$), we obtain a direct product representation of the quantum Galilei group.

We can see this from the viewpoint of cohomological formalism. We construct the quantum mechanical Galilei group, choosing the Galilei transformations

$$\mathbf{r}' = \mathbf{r} + \mathbf{v}t, \qquad \mathbf{p}' = \mathbf{p} + m\mathbf{v} \tag{43}$$

where m is the particle mass.

Then equation (2) is transformed into

$$U(\mathbf{v}) = e^{i\mathbf{v} \cdot (\mathbf{p}t - m\mathbf{r})/\hbar} \tag{44}$$

and its action on a wave function $\Psi(\mathbf{r})$ introduces a phase (one-cocycle) α_1, i.e.,

$$U(\mathbf{v})\cdot\Psi(\mathbf{r}) = e^{2i\pi\alpha_1(\mathbf{r};\mathbf{v})}\cdot\Psi(\mathbf{r} + \mathbf{v}t) \tag{45}$$

We shall consider this one-cocycle as trivial, so

$$\alpha_1(\mathbf{r};\, \mathbf{v}) = \delta\alpha_0 = \alpha_0(\mathbf{r}') - \alpha_0(\mathbf{r}) \tag{46}$$

where α_0 is a function, called the 0-cocycle, which depends only on $\mathbf{r}$.

Therefore, the group law of the quantum mechanical Galilei group for translations on phase space [or $U(1)$ extended Galilei group] is expressed such that

$$e^{2i\pi\alpha_0(\mathbf{r}')} = e^{2i\pi[\alpha_0(\mathbf{r})+\phi+\alpha_1(\mathbf{r};\mathbf{v})]} \tag{47}$$

where

$$2\pi\alpha_1 = \frac{1}{\hbar}\left(m\mathbf{v}\cdot\mathbf{r} + \frac{1}{2}\,mv^2t\right) \tag{48}$$

and ϕ is the central parameter of the quantum mechanical Galilei group.

On the other hand, we require

$$M = \begin{pmatrix} u_1^1 & u_2^1 \\ u_1^2 & u_2^2 \end{pmatrix} \in SL_q(2), \qquad M = \begin{pmatrix} \tilde{u}_1^1 & \tilde{u}_2^1 \\ \tilde{u}_1^2 & \tilde{u}_2^2 \end{pmatrix} \in \tilde{S}L_q(2)$$

to belong to the quantum mechanical Galilei group; i.e., M (equivalently $\tilde{M}$) must fulfill (5). It is straightforward to prove that, in this case, the following null bi-ideals have to be imposed on M (equivalently on $\tilde{M}$)

$$\begin{gathered} u_2^1 = 0, \qquad u_1^2 = 0 \\ u_1^1u_2^2 = u_2^2u_1^1 = 1 \end{gathered} \tag{49}$$

in order to end up with a group that has only one generator, as should be.

In addition, we can prove that the null bi-ideals once imposed on $M \in SL_q(2)$ (thereby defining the quantum mechanical Galilei group) produce the following pairing:

$$\begin{aligned} \langle u_2^1,\, t_m^{\dagger k}\rangle &= R_{2m}^{1k} = 0, \qquad k,\, m,\, s = 1,\, 2 \\ \langle u_1^2,\, t_m^{\dagger k}\rangle &= R_{1m}^{2k} = 0 \\ \langle u_1^1u_2^2,\, t_s^{\dagger k}\rangle &= R_{1m}^{1k}R_{2s}^{2m} = 1 \\ \langle u_2^2u_1^1,\, t_s^{\dagger k}\rangle &= R_{2m}^{2k}R_{1s}^{1m} = 1 \end{aligned} \tag{50}$$

where $t_m^{\dagger k}$ $(k,\, m = 1,\, 2)$ are generators of the dual Hopf algebra for $SL_q(2)$ and R_{kl}^{ij} $(i, j, k, l = 1, 2)$ are entries of the Yang–Baxter matrix R_G associated with the quantum mechanical Galilei group. This does not determine R_G, but restricts the solution to block-diagonal matrices.

Finally, if we impose the null directions given by (50) in Λ, we obtain the following representation of the quantum Galilei group:

$$\Lambda = \begin{pmatrix} (\tilde{u}_1^1)^{-1}u_1^1 & 0 & 0 & 0 \\ 0 & \dfrac{\tilde{u}_1^1 u_1^1 + q^2(\tilde{u}_1^1 u_1^1)^{-1}}{1+q^2} & 0 & \dfrac{q^2(\tilde{u}_1^1 u_1^1 - (\tilde{u}_1^1 u_1^1)^{-1})}{1+q^2} \\ 0 & 0 & \tilde{u}_1^1(u_1^1)^{-1} & 0 \\ 0 & \dfrac{\tilde{u}_1^1 u_1^1 - (\tilde{u}_1^1 u_1^1)^{-1}}{1+q^2} & 0 & \dfrac{q^2(\tilde{u}_1^1 u_1^1 + (\tilde{u}_1^1 u_1^1)^{-1})}{1+q^2} \end{pmatrix} \tag{51}$$

Summarizing, in this paper we have introduced the concept of q-regularization and used the projective representation of the noncommutative Heisenberg algebra to construct the Manin quantum plane, thereby defining q-spinors. Using this as a building block, we first presented a q-regularization in terms of a four-dimensional representation of a particular two-dimensional noncommutative space. We also studied regularization on a q-deformed space that can be mapped into a particular braided product of Manin's quantum plane and related it to the first q-space we studied. We showed how to extract, from relevant quantities, finite components (provided $q \neq 1$) that can become infinite at $q = 1$. To compute the Haar weight, we propose a particular basis projected from q-deformed spaces, so the functions to be q-regularized are to be considered on this frame of reference. An example for $\lambda\phi^4$ field theory was presented. Additional work must be done to generalize our scheme to any arbitrary function on a q-Minkowski spacetime basis.

Finally, in order to learn about the general scheme and its symmetries, we studied the $T = 0$ Haar measure in terms of the $SU_q(2)$ measure and the null directions in the Hopf algebra that lead to a quantum mechanical Galilei group.

Although in this paper we can q-regularize only a class of suitable functions (restricted by the particular basis chosen), we think that the full prescription, derived from physical considerations, might be used to make relevant quantities in field theory finite at $q \neq 1$.

ACKNOWLEDGMENTS

This work was partially supported by CONACYT REF 4336E9406, México. I wish to thank ITD in UC Davis for its hospitality.

REFERENCES

Bonechi, F., Celeghini, E. E., Giachetti, R., Sorace, E., *et al.* (1992a). *Physical Review B,* **46**, 5727.

Bonechi, F., Celeghini, E. E., Giachetti, R., Sorace, E., *et al.* (1992b). *Journal of Physics A,* **25**, L939.

Carow-Watamura, U., Schlieker, M., Scholl, M., and Watamura, S. (1990). *Zeitschrift für Physik C,* **48**, 159.

Carow-Watamura, U., Schlieker, Scholl, M., and Watamura, S. (1991). *International Journal of Modern Physics A,* **6**, 3081.

Connes, A. (1986). *Non-Commutative Differential Geometry,* IHES 62.

Djemai, A. E. F. (1992). *Modern Physics Letters A,* **7**, 3169.

Drinfeld, V. G. (1983a). *Soviet Mathematics Doklady,* **27**, 68.

Drinfeld, V. G. (1983b). *Soviet Mathematics Doklady,* **28**, 667.

Drinfeld, V. G. (1985). *Soviet Mathematics Doklady,* **32**, 254.

Drinfeld, V. G. (1986a). Quantum groups, in *Proceedings of the International Congress of Mathematicians,* Berkeley.

Drinfeld, V. G. (1986b). *Functional Analysis and Its Applications,* **20**, 69.

Fredenhagen, K. (1981). *Communications in Mathematical Physics,* **79**, 141.

Larson, R., and Radford, D. (1988). *American Journal of Mathematics,* **110**, 187.

Majid, S. (1990a). *International Journal of Modern Physics A,* **5**, 4689.

Majid, S. (1990b). *International Journal of Modern Physics A,* **5**, 1.

Manin, Yu. I. (1988). Quantum groups and non-commutative geometry, Centre de Recherches Mathématiques, Université de Montreal.

McKane, A. J. (1980). *Physics Letters A,* **76**, 22.

Parisi, G., and Sourlas, N. (1979). *Physical Review Letters,* **43**, 744.

Parisi, G., and Sourlas, N. (1980). *Journal of Physics Letters (France),* **41**, L403.

Rodríguez-Romo, S. (1994). *International Journal of Theoretical Physics,* **33**, 501.

Woronowicz, S. L. (1987). *Communications in Mathematical Physics,* **111**, 613.